An Analogy Between Heat and Momentum Transfer for the Turbulent Boundary Layer on a Flat Plate

O. T. HANNA and J. E. MYERS

Purdue University, Lafayette, Indiana

The conservation laws for momentum, energy, and mass for turbulent, incompressible boundary-layer flow past a flat plate with constant properties and negligible viscous dissipation may be written as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left[(\nu+E_m)\frac{\partial u}{\partial y}\right] \quad (1)$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=\frac{\partial}{\partial y}\left[(\alpha+E_h)\frac{\partial t}{\partial y}\right] \quad (2)$$

$$u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}=\frac{\partial}{\partial y}\left[(D+E_D)\frac{\partial C}{\partial y}\right] \quad (3)$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (4)$$

The velocities, temperatures, and concentrations are time mean values, and as usual it is assumed that $E_m = E_h = E_D$.

For dilute mixtures with constant wall temperature and concentration the velocity, temperature, and concentration fields are identical (except for a scale factor) when $\nu = \alpha = D$. When this condition is not satisfied however, the simple similarity does not exist. Considering just the energy and diffusion equations one can see that if the solution of either of the equations is known, then the solution of the other may be obtained immediately by substituting into the known solution the diffusivity of the desired field for the diffusivity of the known field. This useful relation between the concentration and temperature fields can be established because the velocity profile is independent of either α or D. However since the velocity profile is usually more likely to be known, an analogous relationship by which the temperature and concentration fields could be determined from the velocity field would be more useful. Unfortunately such a simple relationship does not exist because the velocity field is dependent upon the momentum diffusivity, that is the kinematic viscosity.

The analysis presented here is based on finding an appropriate (fictitious) velocity distribution which is related to the actual temperature profile. This is done in detail for turbulent flow past a flat plate held at a uniform temperature. When this method is applied to the laminar flow situation, the usual well-known results are obtained.

O. T. Manna is at the Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California.

ANALYSIS

It is postulated that the temperature profile is related to the velocity profile corresponding to the fictitious situation of the fluid flowing over a plate and having a free stream velocity U' which is to be determined. In other words $u = f(x, y, U, \nu)$, $u' = f(x, y, U', \nu)$, and $t = g(u')$. The relation between t and u' will be assumed to be linear, and this will be shown to satisfy both differential equations provided U' is chosen properly. Once U' is determined, the temperature profile and the ratio δ'/δ will be known, since the turbulent velocity profile is assumed to be known. This information permits the calculation of the heat transfer coefficient. The Prandtl mixing length assumption that $E_m = 1^2(x, y)\,\partial u/\partial y$ will be used.

The turbulent velocity profile has been represented successfully by

$$\frac{u}{U}=\left(\frac{y}{\delta}\right)^{1/n} \quad (5)$$

This expression is valid throughout the turbulent boundary layer except in a region very close to the wall. It is postulated that $t = C_1u' + C_2$. Differentiating the assumed temperature profile, noting that $E_m = E_h$, and substituting into the energy Equation (2) one gets

$$u\frac{\partial u'}{\partial x}+v\frac{\partial u'}{\partial y}=\frac{\partial}{\partial y}\left[(\alpha+E_m)\frac{\partial u'}{\partial y}\right] \quad (6)$$

In order to proceed further the ratios of u/u' and v/v' are required. When one compares the actual and fictitious velocity distributions

$$\frac{u}{u'}=\frac{U(y/\delta)^{1/n}}{U'(y/\delta')^{1/n}}=\frac{U}{U'}\left(\frac{\delta'}{\delta}\right)^{1/n} \quad (7)$$

But since δ is proportional to $1/U^{1/5}$

$$\frac{u}{u'}=\left(\frac{U}{U'}\right)^{(5n+1)/5n} \quad (8)$$

By differentiating the velocity profile with respect to x and using the continuity Equation (4), one can determine v:

$$v=\frac{4U}{5(n+1)}\frac{y^{(n+1)/n}}{x\,\delta^{1/n}} \quad (9)$$

From this relation the ratio of v/v' is obtained:

$$\frac{v}{v'}=\frac{U}{U'}\left(\frac{\delta'}{\delta}\right)^{1/n}=\frac{u}{u'} \quad (10)$$

It is also necessary to relate E_m and E_m'. Since $E_m = 1^2(x, y)\,\partial u/\partial y$

$$\frac{E_m}{E_m'}=\frac{\partial u/\partial y}{\partial u'/\partial y}=\frac{u/ny}{u'/ny}=\frac{u}{u'} \quad (11)$$

When one assumes u/u' constant

$$\frac{\partial E_m}{\partial y}=\left(\frac{u}{u'}\right)\frac{\partial E'_m}{\partial y} \quad (12)$$

Now by multiplying the transformed energy Equation (6) by u'/u (which equals v'/v) one can see that the left-hand side of the resulting equation is equal to the left-hand side of the momentum Equation (1) written for the fictitious flow situation. Thus equating the right-hand sides of these equations one obtains

$$\left(\frac{u'}{u}\right)\frac{\partial}{\partial y}\left[(\alpha+E_m)\frac{\partial u'}{\partial y}\right]=\frac{\partial}{\partial y}\left[(\nu+E'_m)\frac{\partial u'}{\partial y}\right] \quad (13)$$

Expanding Equation (13), noting that the Prandtl number is to be constant irrespective of the individual variations of ν and α, and using Equations (11) and (12) one gets

$$\nu\frac{\partial^2 u'}{\partial y^2}+\frac{\partial u'}{\partial y}N_{Pr}\frac{\partial \alpha}{\partial y}=\left(\frac{u'}{u}\right)\alpha\frac{\partial^2 u'}{\partial y^2}+\left(\frac{u'}{u}\right)\frac{\partial u'}{\partial y}\frac{\partial \alpha}{\partial y} \quad (14)$$

This relation is seen to be true provided that

$$\frac{u}{u'}=\frac{\alpha}{\nu}=\frac{1}{N_{Pr}}=\text{constant} \quad (15)$$

It has been shown that $t = C_1u' + C_2$, where u' is the longitudinal velocity component of the fluid having a free stream velocity U'. Since u' is zero for $t = t_w$, and $u' = U'$ for $t = t_\infty$

$$\frac{t-t_w}{t_\infty-t_w}=\frac{u'}{U'}=\left(\frac{y}{\delta'}\right)^{1/n} \quad (16)$$

Now the ratio of the thermal to the momentum boundary-layer thickness η will be determined in order to find the local heat transfer coefficient. Using Equations (8) and (15) one obtains

$$\eta\equiv\frac{\delta'}{\delta}=\left(\frac{U}{U'}\right)^{1/5}=\left[\left(\frac{u}{u'}\right)^{5n/(5n+1)}\right]^{1/5}=\left[\frac{1}{N_{Pr}^{5n/(5n+1)}}\right]^{1/5}=\frac{1}{N_{Pr}^{n/(5n+1)}} \quad (17)$$

The integral forms of Equations (1) and (2) are used to determine the local Stanton number (*1*). Using Equations (5) and (16) for the velocity and temperature profiles, respectively, and Equation (17) for η one gets the final expression for the Stanton number:

$$N_{St}\, N_{Pr}^{(n+1)/(\delta_{n+1})} = \frac{f}{2} \qquad (18)$$

DISCUSSION

The same final result can also be derived by relating the heat flux at the wall to the fictitious wall shear stress which can in turn be related to the actual wall shear stress. It should be noted that this type of approach is limited to the case where δ/δ' is greater than unity. This must be the case since only for $\delta \geqq \delta'$ is there a real u for every u' within the fictitious boundary-layer thickness δ'. This means that the Prandtl number must be 1 or greater. The error is small however for fluids like air ($N_{Pr} = .7$).

This analysis assumed the velocity profile and from this derived the temperature profile. It is seen that the two profiles are quite similar, with the temperature profile being characterized by the same exponent n as the velocity profile. This has been borne out experimentally by Reynolds et al. (*2*) who measured velocity and temperature profiles in the turbulent boundary layer. They found the value of the exponent n to be the same for both the velocity and temperature distributions.

As can be seen the Prandtl number dependence of the Stanton number indicated in this analysis is very insensitive to the value of n. For $n = 7$ the exponent is 2/9, while for $n = 5$ it is 3/13.

The only experimental data known to the authors with Prandtl numbers very different from unity are those published recently by Ede and Saunders (*3*). Heat transfer coefficients were compared for water at temperatures such that the Prandtl number was 4.9 and 6.7. Although the Colburn analogy appeared to provide an adequate basis for comparing the effect of Prandtl number variation, scattering of the data make the conclusions somewhat tentative. In addition the effect of an unheated starting length was a part of the investigation, and the effect of this variable complicates the interpretation of the final results.

In a recent investigation Reynolds (*2*) studied incompressible, turbulent boundary layer heat transfer past a flat plate. This work (with air) indicated a Prandtl number dependence of the Stanton number substantially less than the usual Colburn type of correlation. A Prandtl number exponent of 0.4 was used as compared with 0.67 called for by the Colburn analogy.

NOTATION

C = local concentration
C_1, C_2 = constants
C_p = heat capacity
E_D = eddy diffusivity of mass
E_h = eddy diffusivity of heat
E_m = eddy diffusivity of momentum
f = friction factor
h = heat transfer coefficient
k = thermal conductivity
l = Prandtl mixing length
n = exponent in empirical turbulent velocity profile
N_{Pr} = ν/α
$N_{Re\,x}$ = xU/ν
N_{St} = $h/C_p\rho U$
q''_w = wall heat flux
t = local temperature
t_w = wall temperature (constant)
t_∞ = free stream temperature (constant)
u = longitudinal velocity component
u' = fictitious longitudinal velocity component
U = free stream velocity
U' = fictitious free stream velocity
v = transverse velocity component
v' = fictitious transverse velocity component
x = longitudinal distance measured from front of plate
y = transverse distance measured from surface of plate

Greek Letters

α = thermal diffusivity
δ = momentum boundary-layer thickness
δ' = fictitious momentum boundary-layer thickness or actual thermal boundary-layer thickness
η = δ'/δ
ν = kinematic viscosity
ρ = density
τ_w = wall shear stress

LITERATURE CITED

1. Eckert, E. R. G., and R. M. Drake "Heat and Mass Transfer," 2 ed., McGraw-Hill, New York (1959).
2. Reynolds, W. C., W. M. Kays, and S. J. Kline, *Natl. Aeronaut. Space Administration Memo 12-1-58W.*
3. Ede, A. J., and O. A. Saunders, *Proc. Inst. Mech. Engrs.*, **172**, 23 (1958).

CHEMICAL ENGINEERING PROGRESS SYMPOSIUM SERIES ABSTRACTS

The Chemical Engineering Progress Symposium Series is composed of papers on specific subjects conveniently bound in individual books, which are published at intervals. The books are 8½ by 11 inches, paper covered, and cost $4.00 to members, $6.00 to nonmembers for "Heat Transfer—Buffalo," No. 32. They may be ordered from the Secretary's Office, the American Institute of Chemical Engineers, 345 East 47 Street, New York 17, New York.

The *A.I.Ch.E. Journal* will publish, from time to time, abstracts of the articles appearing in the Symposium Series volumes. Recently published volumes are abstracted below.

HEAT TRANSFER—BUFFALO, Vol. 57, No. 32, 1961.

Quantitative Evaluation of the Effect of Edge Losses and Contact Resistances in the Determination of Thermal Diffusivity of Solid Materials by an Unsteady State Method, Arthur A. Armstrong and K. O. Beatty. Thermal diffusivity was determined from time-temperature data. Charts were constructed by relating the temperature at two points in the sample at steady state for various values of the two parameters. The thermal diffusivity was then calculated from the time-temperature data and the values of the two parameters. **On Unsteady State Heat Transfer in a Hollow Cylinder or Sphere**, Warren W. Clauson. This paper describes methods that are useful in the solution of unsteady state heat conduction problems for composite hollow cylinders or spheres. These methods are applicable to problems occurring in many industries; the chemical, missile, aircraft, and power industries are a few examples. A solution may be obtained using graphical or numerical techniques. **Some Aspects of the Melting Solution for a Semi-infinite Slab**, Manfred Altman. It is shown that the general nonlinear problem reduces to a quasilinear one in a region surrounding the peak of the heat-flux curve under certain conditions. Analogue solutions are presented for certain generalized cases. Approximate analytical solutions are presented which are based on the linearized equations resulting from the integral method. **Analysis of Transient Ablation and Heat Conduction Phenomena at a Vaporizing Surface**, R. G. Fleddermann and H. Hurwicz. An exact analysis of heat and mass transfer in an ablating three-phase system is made. Account is also taken of internal heat radiation and of the results of steady state ablation theory. An approximate method based on effective heat of ablation and constant surface temperature is also developed. **Local Radial Effective Conductivity and the Wall Effect in Packed Beds**, R. F. Bad-